# Fourier Series

$$E(A_0, A_1, \cdots, A_N, B_1, \cdots, B_N)$$

$$= \int_0^{2\pi} \left( f(x) - \left( \sum_{n=0}^{N} A_n \cos nx + B_n \sin nx \right) \right)^2 dx$$

$$= \left\langle f(x) - \left( \sum_{n=0}^{N} A_n \cos nx + B_n \sin nx \right), f(x) - \left( \sum_{n=0}^{N} A_n \cos nx + B_n \sin nx \right) \right\rangle$$

$$= \left\| f(x) - \left( \sum_{n=0}^{N} A_n \cos nx + B_n \sin nx \right) \right\|_2^2$$

( $L_2$ - norm between $f(x)$ and $\left( \sum_{n=0}^{N} A_n \cos nx + B_n \sin nx \right)$ ).

Assume $A_0^*, A_1^*, \cdots$ minimizer for $E$.

$j \neq 0$,

$$0 = \frac{\partial E}{\partial A_j}\bigg|_{A_j = A_j^*} = \int_0^{2\pi} -2 \left( f(x) - \left( \sum_{n=0}^{N} A_n \cos nx + B_n \sin nx \right) \right) \cos jx \, dx$$

$$= -2 \Big[ \int_0^{2\pi} f(x) \cos nx \, dx - \int_0^{2\pi} A_j^* \cos jx \cos jx \, dx$$

$$- \int_0^{2\pi} B_j \sin jx \cos jx \, dx$$

$$- \sum_{\substack{n=0 \\ n \neq j}}^{N} \int_0^{2\pi} A_n \cos nx \cos jx \, dx + \int_0^{2\pi} B_n \sin nx \cos jx \, dx \Big]$$

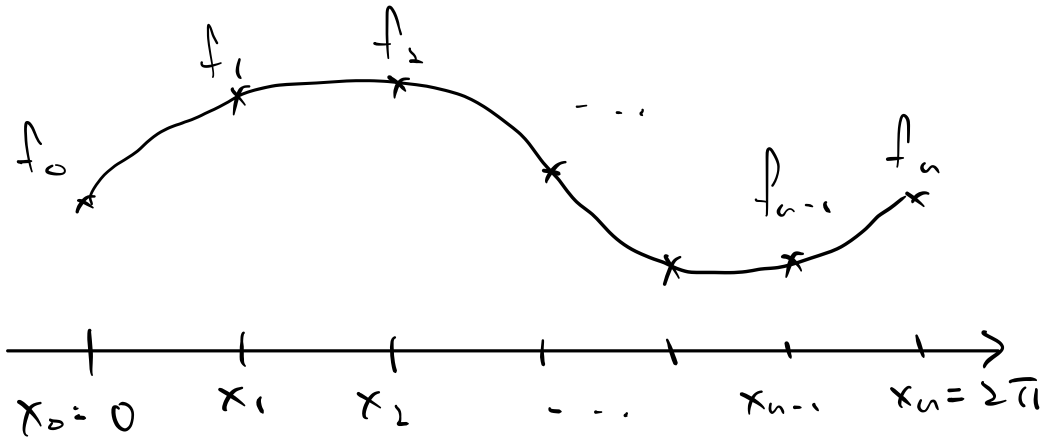
$$\Rightarrow A_j^* = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos jx \, dx.$$

$$\left.\frac{\partial^2 E}{\partial A_j^2}\right|_{A_j = A_j^*} = -2\int_0^{2\pi} -(\cos jx)^2 \, dx > 0.$$

$\therefore$ $A_j^*$ is minimizer.

Similar Approach works for $A_0^*$ and $B_n^*$.

# Discrete Fourier Transform




$$\vec{f} = [f_0, f_1, \cdots f_{n-1}]^T \in \mathbb{C}^n.$$

$$\text{DFT:} \quad \vec{c} = \frac{1}{n} \overline{A_\omega}\, \vec{f}$$

$$\text{IDFT:} \quad \vec{f} = A_\omega \vec{c} = \begin{bmatrix} | & | & & | \\ \overrightarrow{e^{i(0)x}} & \overrightarrow{e^{i(1)x}} & \cdots & \overrightarrow{e^{i(n-1)x}} \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

$$= c_0 \overrightarrow{e^{i(0)x}} + c_1 \overrightarrow{e^{i(1)x}} + \cdots + c_{n-1} \overrightarrow{e^{i(n-1)x}}$$

$$\text{where} \quad \overrightarrow{e^{imx}} = \begin{bmatrix} e^{imx_0} \\ e^{imx_1} \\ \vdots \\ e^{imx_{n-1}} \end{bmatrix} = \begin{bmatrix} e^{2\pi i m(0)/n} \\ e^{2\pi i m(1)/n} \\ \vdots \\ e^{2\pi i m(n-1)/n} \end{bmatrix}$$

# Numerical Spectral Method.

$Lu = f \quad , \quad x \in [0, 2\pi]. \quad u(0) = u(2\pi).$

$L$: linear differential operator.

Don't know $f$,

but only $\vec{f}$, $n$ measurements of $f$.

Assume $f$ is $2\pi$ - periodic

Then we have $f_k = f_{k+n} = f_{k+2n} = \cdots$

The periodicity is very Importance.

Idea:

Discretized the ODE:

$$Lu = f \Rightarrow \tilde{D}\vec{u} = \vec{f}$$

Express $\vec{u}$, $\vec{f}$ in basis vector $\{\vec{\phi}_j\}_{j=0}^{n-1}$

where $\{\vec{\phi}_j\}_{j=0}^{n-1}$ are eigenvectors of $\tilde{D}$.

Then $\tilde{D}\vec{u} = \vec{f}$

$$\Rightarrow \tilde{D}\sum_{j=0}^{n-1}\hat{u}_j\vec{\phi}_j = \sum_{j=0}^{n-1}\hat{f}_j\vec{\phi}_j$$

$$\Rightarrow \sum_{j=0}^{n-1}\hat{u}_j\left(\tilde{D}\vec{\phi}_j\right) = \sum_{j=0}^{n-1}\hat{f}_j\vec{\phi}_j$$

$$\Rightarrow \sum_{j=0}^{n-1}\hat{u}_j\lambda_j\vec{\phi}_j = \sum_{j=0}^{n-1}\hat{f}_j\vec{\phi}_j$$

$$\therefore \hat{u}_j\lambda_j = \hat{f}_j$$

Recover $\vec{u}$ by $\sum_{j=0}^{n-1}\hat{u}_j\vec{\phi}_j$.

Example.

$$\frac{du}{dx} = f$$

First, discretize $\frac{d}{dx}$:

Recall $$\frac{du}{dx} = \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

$$\approx \frac{u(x+h) - u(x)}{h}$$

$$\tilde{D} = \frac{1}{2\pi/n} \begin{bmatrix} -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \\ 1 & & & -1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Then $\frac{du}{dx} = f$ can be discretized as:

$$\tilde{D} \vec{u} = \vec{f}$$

Note by IDFT:

$$\tilde{D} \left( \sum_{k=0}^{n-1} \hat{u}_k \overrightarrow{e^{ikx}} \right) = \sum_{k=0}^{n-1} \hat{f}_k \overrightarrow{e^{ikx}}$$

where $\hat{u} = \begin{bmatrix} \hat{u}_0 \\ \hat{u}_1 \\ \vdots \\ \hat{u}_{n-1} \end{bmatrix}$, $\hat{f} = \begin{bmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \vdots \\ \hat{f}_{n-1} \end{bmatrix}$

are DFT of $\vec{u}$ and $\vec{f}$.

$$\Rightarrow \sum_{k=0}^{n-1} \hat{u}_k \left( \tilde{D} \, \overrightarrow{e^{ikx}} \right) = \sum_{k=0}^{n-1} \hat{f}_k \, \overrightarrow{e^{ikx}}$$

$\vdash$: $\overrightarrow{e^{ikx}}$ is a eigenvector of $\tilde{D}$.

$k = 0, \cdots, n-1$.

$j = 1, 2, \cdots, n$

j-th entry of $\overrightarrow{e^{ikx}}$ : $\left( \overrightarrow{e^{ikx}} \right)_j = e^{2\pi i k (j-1)/n}$

j-th entry of $\tilde{D} \, \overrightarrow{e^{ikx}}$ :

$$\left( \tilde{D} \, \widetilde{e^{ikx}} \right)_j = \frac{n}{2\pi} \left( e^{2\pi i k (j)/n} - e^{2\pi i k (j-1)/n} \right)$$

$$= \frac{n}{2\pi} \left( e^{2\pi i k/n} - 1 \right) e^{2\pi i k (j-1)/n}$$

$$= \frac{n}{2\pi} \left( e^{2\pi i k/n} - 1 \right) \left( \overrightarrow{e^{ikx}} \right)_j$$

$\therefore$ $\overrightarrow{e^{ikx}}$ is a eigenvector of $\tilde{D}$

with eigenvalue $= \frac{n}{2\pi} \left( e^{2\pi i k/n} - 1 \right)$.

$$\sum_{k=0}^{n-1} \hat{u}_k \left( \tilde{D} \vec{e^{ikx}} \right) = \sum_{k=0}^{n-1} \hat{f}_k \vec{e^{ikx}}$$

$$\Rightarrow \sum_{k=0}^{n-1} \hat{u}_k \left( \lambda_k \right) \vec{e^{ikx}} = \sum_{k=0}^{n-1} \hat{f}_k \vec{e^{ikx}}$$

$\left\{ \vec{e^{ikx}} \right\}_{k=0}^{n-1}$ Linearly Independent

$$\Rightarrow \hat{u}_k \lambda_k = \hat{f}_k .$$

Reduced ODE to Algebraic Equation.

Note the discretization of $\frac{d}{dx}$ is not unique.

By Taylor Expansion:

$$(1) - u(x+h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2} + \cdots$$

$$(2) - u(x-h) = u(x) - u'(x)h + u''(x)\frac{h^2}{2} + \cdots$$

$$(1) \Rightarrow \frac{u(x+h) - u(x)}{h} = u'(x) + u''(x)\frac{h}{2} + \cdots$$

$\therefore \frac{u(x+h)-u(x)}{h}$ approximate $u'(x)$

with error $O(h)$.

$[(1) - (2)]/2h$:

$$\frac{u(x+h) - u(x-h)}{2h} = u'(x) + u'''(x)\frac{h^2}{12} + \cdots$$

$\therefore \frac{u(x+h)-u(x-h)}{2h}$ approximate $u'(x)$

with error $O(h^2)$.

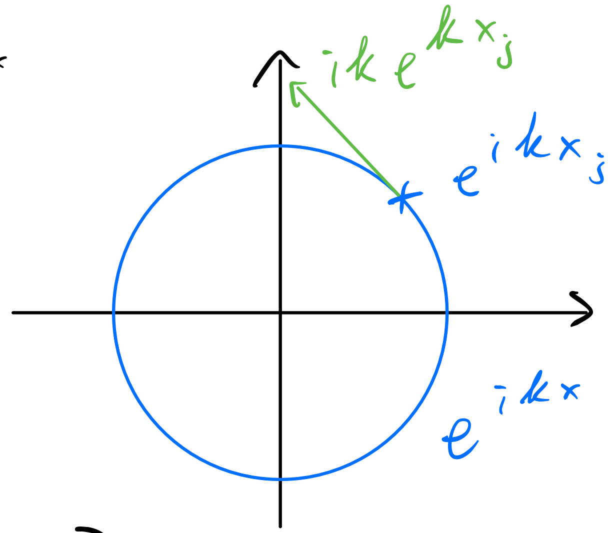
$$\therefore \tilde{D} = \frac{n}{\pi}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \\ 1 & & & -1 & 0 \end{bmatrix}$$ is an

approximation with less error.

Graphical Illustration for the discretizations:

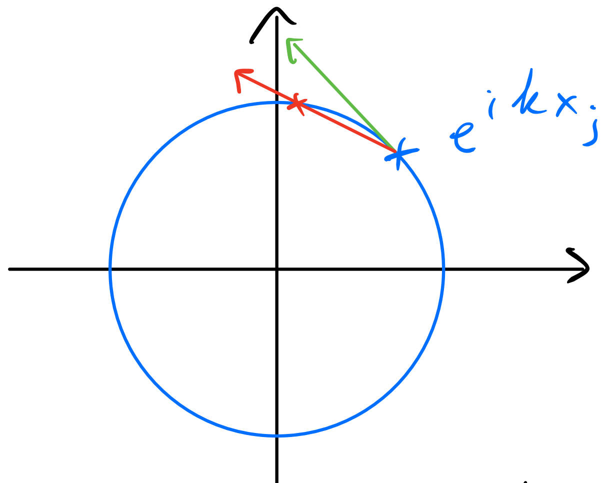
True function $f$: $e^{ikx}$

Derivative: $ike^{ikx}$

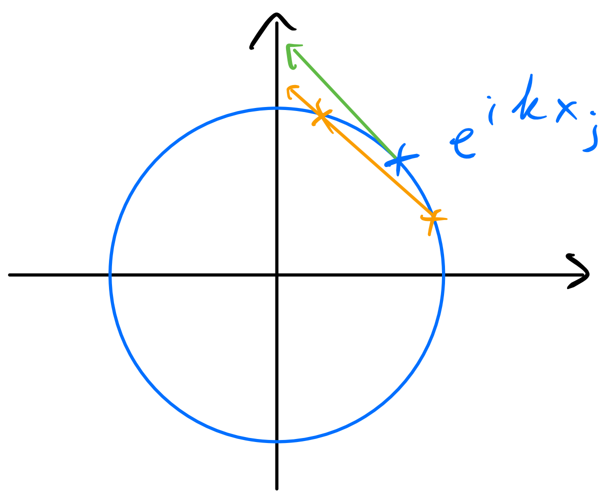



Discretization for $f$: $\overrightarrow{e^{ikx}}$

Discretization 1 for derivative: $\dfrac{e^{ikx_{j+1}} - e^{ikx_j}}{h}$




Discretization 2 for derivative: $\dfrac{e^{ikx_{j+1}} - e^{ikx_{j-1}}}{2h}$



## Exercise:

(1). Show that

$$\tilde{D} = \frac{n}{10\pi}\begin{bmatrix} 0 & 1 & 1 & & & -2 \\ -2 & 0 & 1 & 1 & & \\ & & & \ddots & & \\ 1 & 1 & & & -2 & 0 \end{bmatrix}$$

is a discretization for $\frac{d}{dx}$.

(2). Show that

$\vec{e^{ikx}}$ is a eigenvector for $\tilde{D}$.

Exercise Solution.

(1) — $u(x+h) = u(x) + u'(x)h + u''(x)\frac{h^2}{2} + \cdots$

(2) — $u(x+2h) = u(x) + u'(x)(2h) + u''(x)\frac{(2h)^2}{2} + \cdots$

(3) — $u(x-h) = u(x) - u'(x)h + u''(x)\frac{h^2}{2} + \cdots$

(1) + (2) − (3) × 2

$\Rightarrow \quad u(x+h) + u(x+2h) - 2u(x-h) = u'(x)(5h) + \cdots$

$$\therefore \quad u'(x) \approx \frac{u(x+h) + u(x+2h) - 2u(x-h)}{5h}$$

$$\overrightarrow{e^{ikx}} = \begin{bmatrix} \left(\overrightarrow{e^{ikx}}\right)_1 \\ \left(\overrightarrow{e^{ikx}}\right)_2 \\ \vdots \\ \left(\overrightarrow{e^{ikx}}\right)_n \end{bmatrix}, \quad \tilde{D}\,\overrightarrow{e^{ikx}} = \begin{bmatrix} \left(\tilde{D}\,\overrightarrow{e^{ikx}}\right)_1 \\ \left(\tilde{D}\,\overrightarrow{e^{ikx}}\right)_2 \\ \vdots \\ \left(\tilde{D}\,\overrightarrow{e^{ikx}}\right)_n \end{bmatrix}$$

j-th entry of $\overrightarrow{e^{ikx}}$

$$\underline{\left(\overrightarrow{e^{ikx}}\right)_j} = e^{2\pi i k(j-1)/n}$$

$$\left(\tilde{D}\,\overrightarrow{e^{ikx}}\right)_j = \frac{n}{10\pi}\left[e^{2\pi i k(j+1)/n} + e^{2\pi i k j/n} - 2e^{2\pi i(j-2)/n}\right]$$

$$= \underbrace{\frac{n}{10\pi}\left[e^{4\pi i k/n} + e^{2\pi i k/n} - 2e^{-2\pi i k/n}\right]}_{\lambda_k} e^{2\pi i k(j-1)/n}$$

$$= \lambda_k \left(\overrightarrow{e^{ikx}}\right)_j$$

$$\tilde{D}\,\overrightarrow{e^{ikx}} = \begin{bmatrix} (\tilde{D}\,\overrightarrow{e^{ikx}})_1 \\ (\tilde{D}\,\overrightarrow{e^{ikx}})_2 \\ \vdots \\ (\tilde{D}\,\overrightarrow{e^{ikx}})_n \end{bmatrix} = \begin{bmatrix} \lambda_k(\overrightarrow{e^{ikx}})_1 \\ \lambda_k(\overrightarrow{e^{ikx}})_2 \\ \vdots \\ \lambda_k(\overrightarrow{e^{ikx}})_n \end{bmatrix}$$

$$= \lambda_k \begin{bmatrix} (\overrightarrow{e^{ikx}})_1 \\ (\overrightarrow{e^{ikx}})_2 \\ \vdots \\ (\overrightarrow{e^{ikx}})_n \end{bmatrix} = \lambda_k\,\overrightarrow{e^{ikx}}$$